A GLOBAL CONVOLUTION DIAGRAM FOR $\mathcal{R}$

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Abstract. The aim of this appendix is to give another proof of the commutativity of the Coulomb branch by constructing a global convolution diagram for $\mathcal{R}$. This is a direct generalization of the traditional proof of the case $\mathbf{N} = 0$, which uses the Beilinson-Drinfeld global convolution diagram for Gr.

1. Preliminaries on arc-spaces and loop-spaces

1.1. In this section, we recall certain standard constructions and facts of [3], Chapters 4-5.

1.2. Let X be a smooth complex curve and let S be a finite set. Given a commutative ring R and an R-point x of X^S, we denote the coordinates of x by x_s ($s \in S$), and write $\Delta_S(x)$ for the formal neighborhood of the union of the graphs of x_s ($s \in S$). For notational simplicity, we frequently remove commas and braces from S, and also drop the part (x), when it is clear which point we refer to. So for example the expression:

$$\Delta_{\{1,2\}}(x)$$

becomes:

$$\Delta_{12}.$$

1.3. Now fix an affine algebraic group A over $\mathbb{C}$. Consider the following functor from commutative rings to groups over X^S:

$$A_S(R) := \{(x, f) | x \in X^S(R), f : \Delta_S \to A\}.$$

Then A_S is represented by the limit of a projective system of smooth affine group schemes over X^S:

$$A_S = \varprojlim(\ldots \to (A_S)_2 \to (A_S)_1)$$

such that each transition morphism is a smooth homomorphism. In particular, A_S is a formally smooth affine group scheme (of countably infinite type) over X^S, but this is not so important for us. Recall that in the definition of the Coulomb branch as a convolution algebra formal homological shifts such as

$$[2 \dim A(\mathcal{O})]$$

appear (for $A = G, N$). Similarly, in the global situation formal homological shifts such as

$$[2 \dim A_S]$$

will appear[1]. For example, in the case where the underlying space is A_S, for each d we have $\omega_{(A_S)_d} \cong \mathbb{C}_{(A_S)_d}[2 \dim(A_S)_d]$. These complexes are compatible in the

[1] Only for S of cardinality 1 or 2; but it clarifies the picture and simplifies the exposition to work more generally at this point.

natural way under !-pullbacks along the transition morphisms. We thus consider ω_{A_S} as the formal homological shift

$$\omega_{A_S} \cong \mathbb{C}_{A_S}[2\dim A_S],$$

where both sides are to be understood by evaluating on smooth quotients of A_S and 'piecing together' using !-pullbacks. Likewise we have a formal expression

$$\omega_{A_S}[-2\dim A_S] \cong \mathbb{C}_{A_S}$$

where both sides are to be understood by evaluating on the smooth quotients $(A_S)_d$ of A_S and 'piecing together' using $*$-pullbacks.

1.4. Let $\theta : S' \to S$ be a morphism of finite sets. It induces a map $X^S \to X^{S'}$. Given an R-point x of X^S, this map determines an R-point x' of $X^{S'}$, and an embedding $\Delta_{S'}(x') \to \Delta_S(x)$. Hence by restriction along this embedding we obtain a map

$$p^\theta : A_S \to A_{S'}.$$

This induces a homomorphism

$$q^\theta : A_S \to A_{S'} \times_{X^{S'}} X^S$$

over the base X^S. If θ is surjective, then q^θ is an isomorphism. If θ is injective, then q^θ seems strange at first sight. For instance if θ' is a section of θ then q^θ is an isomorphism over the resulting copy of $X^{S'} \subset X^S$, whereas over a typical point of X^S, q^θ takes the form of a projection map

$$A(\mathcal{O})^S \to A(\mathcal{O})^{S'}.$$

However, this is misleading: q^θ is *pro-smooth* when θ is injective. What we mean by this is that the projective systems of smooth affine group schemes over X^S with smooth transition morphisms

$$((A_S)_d)_{d\in\mathbb{N}}$$

underlying A_S may be taken, simultaneously for all S, to be compatible with all q^θ, i.e. so that q^θ is the limit of a morphism

$$(q^\theta_d : (A_S)_d \to (A_{S'})_d \times_{X^{S'}} X^S)_{d\in\mathbb{N}}$$

of projective systems, where each map q^θ_d is a smooth homomorphism over X^S. Thus, it makes sense to write (and is true that):

$$(p^\theta)^*\omega_{A_{S'}\times_{X^{S'}}(X^S)}[-2\dim A_{S'} - 2(|S| - |S'|)] = \omega_{A_S}[-2\dim A_S],$$

et cetera, where the formula should be understood as a statement about complexes on smooth quotients over X^S, compatible under $*$-pullbacks.

Example 1.1. *Consider the case $S = \{1\}$. Then A_1 is a Zariski-locally trivial $A(\mathcal{O})$-bundle over X. Then, the formal homological shift $[2\dim A(\mathcal{O})]$ also makes sense in this context, and we have $[2\dim A(\mathcal{O})] = [2\dim A_1 - 2]$.*

Example 1.2. *Consider for instance the case $A = \mathbb{C}$ and $S = \{1, 2\}$. Then A_{12} should be thought of as a deformation of the first following projective system into the second:*

$$(\mathbb{C}[[t]]/t^{2d})_d \quad \rightsquigarrow \quad (\mathbb{C}[[t]]/t^d \times \mathbb{C}[[t]]/t^d)_d$$

while A_1 should be thought of as a trivial deformation:

$$(\mathbb{C}[[t]]/t^d)_d \quad \rightsquigarrow \quad (\mathbb{C}[[t]]/t^d)_d$$

and we have the morphism of deformations of projective systems:

$$\begin{array}{ccc} (\mathbb{C}[[t]]/t^{2d})_d & \rightsquigarrow & (\mathbb{C}[[t]]/t^d \times \mathbb{C}[[t]]/t^d)_d \\ \downarrow & & \downarrow \\ (\mathbb{C}[[t]]/t^d)_d & \rightsquigarrow & (\mathbb{C}[[t]]/t^d)_d \end{array}$$

where the first downward arrow is the quotient map, and the second downward arrow is the projection map (to the first factor), both of which halve dimension in the d^{th} approximation. It just happens that the limit of the first downward arrow is an isomorphism, while the limit of the second downward arrow is a non-trivial projection.

1.5. From now on, we assume θ is an injection, and identify S' with its image under θ. Now, in addition to the formal neighborhood Δ_S we have the punctured formal neighborhood

$$\Delta_S^{S'}(x) := \Delta_S(x) - \cup_{s\in S'} x_s$$

where in this formula we conflate the point x_s with its graph. The general notational paradigm[2] here is that subscripts determine discs and superscripts determine punctures. Consider the functor

$$A_S^{S'}(R) := \{(x,f) | x \in X^S(R), f : \Delta_S^{S'}(x) \to A\}.$$

Then $A_S^{S'}$ is represented by an ind-scheme, formally smooth over X^S. It is a group in ind-schemes (over X^S), but not an inductive limit of groups. Nonetheless, it is an *ind-locally nice, reasonable* ind-scheme in the sense of [2], meaning that it is a direct limit of closed embeddings with finitely generated ideals:

$$(A_S^{S'})^1 \to (A_S^{S'})^2 \to \dots$$

of schemes over X^S, each of which is locally nice, meaning that Zariski-locally[3] it is the product of a finite-type scheme with an affine space (of countable dimension). We shall call such an ind-scheme *reasonably nice.* The subgroup A_S may be taken as the first subscheme $(A_S^{S'})^1$ in this inductive structure. The left- and right-regular actions of the subgroup A_S preserve the inductive structure, meaning that each $(A_S^{S'})^c$ has an action on both sides by A_S over X^S, even though it is not itself a group. Moreover the quotient $(A_S^{S'})^c/A_S$ is of finite-type over X^S, and flat, although generally quite singular. The result is that the quotient

$$A_S^{S'}/A_S$$

has the structure of ind-finite-type flat ind-scheme over X^S.

Lemma 1.3. (1) *$A_S^{S'}/A_S$ is ind-projective if and only if A is reductive.*
(2) *$A_S^{S'}/A_S$ is reduced if and only if A has non no-trivial characters.*

Remark 1.4. A_S^S is the Beilinson-Drinfeld grassmannian (on $|S|$ points).

[2]Warning: this doesn't apply to X!
[3]In [2] this is relaxed to 'Nisnevich-locally'.

1.6. For any chain of inclusions $S'' \xrightarrow{\theta'} S' \xrightarrow{\theta} S$ we have natural maps

$$p^\theta : A_S^{S''} \to A_{S'}^{S''},$$

$$q^\theta : A_S^{S''} \to A_{S'}^{S''} \times_{X^{S'}} X^S,$$

defined as in subsection 1.4. Then $q^\theta : A_S^{S''} \to A_{S'}^{S''} \times_{X^{S'}} X^S$ has as a subgroup $q^\theta : A_S \to A_{S'} \times_{X^{S'}} X^S$, and the resulting map

$$A_S^{S''}/A_S \to (A_{S'}^{S''}/A_{S'}) \times_{X^{S'}} X^S$$

is an isomorphism.

Warning 1.5. Observe that $A_S^{S''}$ is an ind-A_S-torsor over the ind-scheme $(A_{S'}^{S''}/A_{S'}) \times_{X^{S'}} X^S$, and the homomorphism $q^\theta : A_S^{S''} \to A_{S'}^{S''} \times_{X^{S'}} X^S$ is surjective. It is tempting therefore to try to view $A_S^{S''}$ as being in some sense a torsor over $A_{S'}^{S''} \times_{X^{S'}} X^S$ for some group $\ker q^\theta$. However, the kernel of the projective system

$$((A_S)_d \to (A_{S'})_d \times_{X^{S'}} X^S)_{d\in\mathbb{N}}$$

of subsection 1.4 is not Mittag-Leffler. We are not sure how to overcome this issue, so do not attempt to take this point of view.

2. Global convolution diagram for $\mathcal{R}$

2.1. For a finite set S, we put

$$\mathcal{T}_S^{S'}(R) = \{(x, \mathcal{E}, f, \tilde{v})\}/\sim$$

where $x \in X^S(R)$, $\mathcal{E}$ is a principal G-bundle on Δ_S, f is a trivialization of $\mathcal{E}$ on $\Delta_S^{S'}$, and $\tilde{v}$ is an $\mathbf{N}$-section of $\mathcal{E}$, taken up to equivalence. This is the same as the balanced product

$$\mathcal{T}_S^{S'} = G_S^{S'} \frac{\times_{X^S}}{G_S} \mathbf{N}_S\,.$$

Thus, $\mathcal{T}_S^{S'}$ is represented by a reasonably nice ind-scheme with an ind-pro-smooth map to the Beilinson-Drinfeld grassmannian $G_S^{S'}/G_S$. In particular it is formally smooth. Multiplication gives us a map

$$\mathcal{T}_S^{S'} \to \mathbf{N}_S^{S'}$$

and we define $\mathcal{R}_S^{S'}$ to be the fiber product

$$\mathcal{R}_S^{S'} := \mathcal{T}_S^{S'} \times_{\mathbf{N}_S^{S'}} \mathbf{N}_S\,.$$

Over any closed X^S-subscheme of $G_S^{S'}/G_S$, the embedding $\mathcal{R}_S^{S'} \to \mathcal{T}_S^{S'}$ has finite codimension. Therefore $\mathcal{R}_S^{S'}$ is also a reasonably nice ind-scheme, mapping to $G_S^{S'}/G_S$, and of ind-finite codimension in $\mathcal{T}_S^{S'}$. Note that $\mathcal{R}_S^{S'}$ is *not* formally smooth, and in particular the map $\mathcal{R}_S^{S'} \to G_S^{S'}/G_S$ is no longer ind-pro-smooth. As a functor we have

$$\mathcal{R}_S^{S'}(R) = \{(x, \mathcal{E}, f, v)\}/\sim$$

where $x, \mathcal{E}, f$ are as in $\mathcal{T}_S^{S'}$, and v is an $\mathbf{N}$-section of $\mathcal{E}$ such that $f(v)$ extends[4] to Δ_S. We define the shifted dualizing complex on $\mathcal{T}_S^{S'}$, $\mathcal{R}_S^{S'}$ as for $\mathcal{T}$, $\mathcal{R}$. Namely:

[4]It is a priori defined only on $\Delta_S^{S'}$. The extension is necessarily unique.

(1) On each closed subscheme $(\mathcal{T}_S^{S'})^c$ of $(\mathcal{T}_S^{S'})^c$, pro-smooth over $(G_S^{S'}/G_S)^c$ we set

$$\omega_{(\mathcal{T}_S^{S'})^c}[-2\dim \mathbf{N}_S + 2|S|]$$

to be the pullback of the dualizing complex of $(G_S^{S'}/G_S)^c$, i.e. the collection of its pullbacks to each formally smooth quotient $(\mathcal{T}_S^{S'})^c_d$ of $(\mathcal{T}_S^{S'})^c_d$ smooth over $(G_S^{S'}/G_S)^c$, compatible under $*$-pullback;

(2) Since $\mathcal{T}_S^{S'}$ is a reasonably nice ind-scheme, we can apply the !-pullback to such a collection of complexes on $(\mathcal{T}_S^{S'})^c$, and obtain one on $(\mathcal{T}_S^{S'})^{c-1}$. In this way, the collections $\omega_{(\mathcal{T}_S^{S'})^c}[-2\dim \mathbf{N}_S + 2|S|]$ are compatible under !-pullbacks. The resulting compatible collection is called $\omega_{\mathcal{T}_S^{S'}}[-2\dim \mathbf{N}_S + 2|S|]$.

(3) Using the ind-finite codimensionality of the embedding $i : \mathcal{R}_S^{S'} \to \mathcal{T}_S^{S'}$, we form a !-compatible collection of $*$-compatible collections of complexes

$$\omega_{\mathcal{R}_S^{S'}}[-2\dim \mathbf{N}_S + 2|S|] := i^! \omega_{\mathcal{T}_S^{S'}}[-2\dim \mathbf{N}_S + 2|S|].$$

2.2. We will apply the abbreviations of subsection 1.2 to our spaces $\mathcal{R}, \mathcal{T}$ etc. so that for instance

$$\mathcal{R}_{\{1,2\}}^{\{2\}}$$

becomes

$$\mathcal{R}_{12}^2 .$$

We will also write X^S as $\Pi_{s\in S} X_s$, e.g. $X^{\{1,2\}} = X_1 \times X_2$. The obvious starting point for the global convolution diagram is $\mathcal{R}_1^1 \times \mathcal{R}_2^2$, a Zariski-locally trivial $\mathcal{R}$-bundle over $X_1 \times X_2$. Consider the following space:

$$\mathcal{R}_{1+2}(R) = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2)\}/\sim$$

where x_1, x_2 are R-points of X, each $\mathcal{E}_i$ a principal G-bundle on Δ_{12}, f_i is a trivialization of $\mathcal{E}_i$ on Δ_{12}^i, and v_i is an N-section of $\mathcal{E}_i$ such that $f_i(v_i)$ extends to Δ_{12}. It is constructed as

$$\mathcal{R}_{1+2} = \mathcal{R}_{12}^1 \times_{X_1\times X_2} \mathcal{R}_{12}^2,$$

a reasonably nice ind-scheme over $X_1 \times X_2$. It is of ind-finite codimension in the formally smooth reasonably nice ind-scheme

$$\mathcal{T}_{1+2} = \mathcal{T}_{12}^1 \times_{X_1\times X_2} \mathcal{T}_{12}^2 = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, \tilde{v}_1, \tilde{v}_2)\}/\sim .$$

There is a map

$$\alpha : \mathcal{R}_{1+2} \to \mathcal{R}_1^1 \times \mathcal{R}_2^2$$

given by restricting $\mathcal{E}_i, f_i, v_i$ to $\Delta_i \subset \Delta_{12}$. Over the diagonal $X \subset X_1 \times X_2$, this map α is an isomorphism. But on the complement U of the diagonal, we have a canonical isomorphism

$$\mathcal{R}_{1+2}|_U = (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U \times_U (\mathbf{N}_1 \times \mathbf{N}_2)|_U$$

and α is just the projection. Nonetheless, *α is ind-pro-smooth.* Indeed, it is the product over $X_1 \times X_2$ of maps

$$\mathcal{R}_{12}^1 \to \mathcal{R}_1^1 \times X_2,$$

$$\mathcal{R}_{12}^2 \to \mathcal{R}_2^2 \times X_1;$$

so it suffices to see that the former is ind-pro-smooth. But note that we can write

$$\mathcal{T}_1^1 \times X_2 = G_1^1 \frac{\times_{X_1}}{G_1} \mathbf{N}_1 \times X_2 = G_{12}^1 \frac{\times_{X_1 \times X_2}}{G_{12}} \mathbf{N}_1$$

where G_{12} acts on $\mathbf{N}_1$ via the homomorphism $G_{12} \to G_1$. Then, the natural map

$$\mathcal{T}_{12}^1 \to \mathcal{T}_1^1 \times X_2$$

is that associated to the pro-smooth map $\mathbf{N}_{12} \to \mathbf{N}_1$, so is ind-pro-smooth. The fact that the diagram

$$\begin{array}{ccc} \mathcal{R}_{12}^1 & \to & \mathcal{R}_1^1 \times X_2 \\ \downarrow & & \downarrow \\ \mathcal{T}_{12}^1 & \to & \mathcal{T}_1^1 \times X_2 \end{array}$$

is Cartesian gives the result. We have:

$$\alpha^* \omega_{\mathcal{R}_1^1 \times \mathcal{R}_2^2}[-2 \dim \mathbf{N}_1 \times \mathbf{N}_2] \cong \omega_{\mathcal{R}_{1+2}}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}]. \tag{2.1}$$

Note that $\mathcal{R}_1^1 \times \mathcal{R}_2^2$, $\mathcal{T}_1^1 \times \mathcal{T}_2^2$ are acted on factor-wise by $G_1 \times G_2$, which receives the factor-wise map from $G_{12} \times_{X_1 \times X_2} G_{12}$. This latter group also acts in the natural way on $\mathcal{R}_{1+2}$, $\mathcal{T}_{1+2}$, and the diagram

$$\begin{array}{ccc} \mathcal{R}_{1+2} & \to & \mathcal{R}_1^1 \times \mathcal{R}_2^2 \\ \downarrow & & \downarrow \\ \mathcal{T}_{1+2} & \to & \mathcal{T}_1^1 \times \mathcal{T}_2^2 \end{array}$$

is $G_{12} \times_{X_1 \times X_2} G_{12}$-equivariant. This action preserves the inductive structure of the diagram, and also the locally nice structure of each closed piece, which allows us to view the appropriately shifted dualizing complex on each space as $G_{12} \times_{X_1 \times X_2} G_{12}$-equivariant. We may thus define the shifted equivariant Borel-Moore homologies:

$$H_{*-2\dim \mathbf{N}_1 \times \mathbf{N}_2}^{BM, G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2),$$

$$H_{*-2\dim \mathbf{N}_1 \times \mathbf{N}_2}^{BM, G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_1^1 \times \mathcal{R}_2^2),$$

$$H_{*-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}^{BM, G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_{1+2}),$$

as the colimits of the equivariant cohomologies of the appropriately shifted dualizing complexes on the various finite-dimensional approximations. We have maps

$$H_{*-2\dim \mathbf{N}_1 \times \mathbf{N}_2}^{BM, G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) \to H_{*-2\dim \mathbf{N}_1 \times \mathbf{N}_2}^{BM, G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) \to H_{*-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}^{BM, G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_{1+2}).$$

The first map is the restriction of the equivariant structure, while the second is induced by α^*, using equation 2.1. This is the first step of our global convolution story.

2.3. Let's define the remaining parts of the global convolution diagram. We set

$$\widetilde{\mathcal{R}}_{1+2} = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2, g_1)\}/\sim$$

where $x_1, x_2, \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2$ are as in $\mathcal{R}_{1+2}$, and g_1 is a trivialization of $\mathcal{E}_1$ (on Δ_{12}) required to satisfy:

$$g_1 v_1 = f_2 v_2.$$

Note that v_1 is determined by the rest of the data as $v_1 = g_1^{-1} f_2 v_2$. That is, $\widetilde{\mathcal{R}}_{1+2}$ is related to

$$\widetilde{\mathcal{T}}_{1+2} := \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_2, g_1)\}/\sim = G_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2$$

by the Cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{R}}_{1+2} & \to & \widetilde{\mathcal{T}}_{1+2} \\ \downarrow & & \downarrow \\ \mathcal{R}^1_{12} & \to & \mathcal{T}^1_{12} \end{array}$$

where the rightmost downward arrow is the composition

$$\widetilde{\mathcal{T}}_{1+2} = G^1_{12} \times_{X_1 \times X_2} \mathcal{R}^2_{12} \to G^1_{12} \times_{X_1 \times X_2} \mathbf{N}_{12} \to G^1_{12} \frac{\times_{X_1 \times X_2}}{G_{12}} \mathbf{N}_{12} = \mathcal{T}^1_{12}\,.$$

We have factor-wise actions of $G_{12} \times_{X_1 \times X_2} G_{12}$ on $\widetilde{\mathcal{R}}_{1+2}$, $\widetilde{\mathcal{T}}_{1+2}$, such that the Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{R}}_{1+2} & \stackrel{\beta}{\to} & \mathcal{R}_{1+2} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{T}}_{1+2} & \stackrel{b}{\to} & \mathcal{T}^1_{12} \times_{X_1 \times X_2} \mathcal{R}^2_{12} \end{array} \tag{2.2}$$

is equivariant. In terms of points, the left-hand G_{12} acts by changing the trivialization f_1, while the right-hand factor acts by changing simultaneously the trivializations g_1, f_2; β is the map which simply forgets g_1. The right-hand G_{12} acts freely, and the quotient space is

$$\overline{\mathcal{R}}_{1+2} = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, g_1^{-1} f_2, v_1, v_2)\}/\sim$$

where $x_1, x_2, \mathcal{E}_1, \mathcal{E}_2, f_1, v_1$ are as in $\mathcal{R}_{1+2}$, while $g_1^{-1} f_2$ is an isomorphism from $\mathcal{E}_2$ to $\mathcal{E}_1$ over Δ^2_{12}, and v_2 is an N-section of $\mathcal{E}_2$ such that $g_1^{-1} f_2 v_2$ extends to Δ_{12} and is equal to v_1 there (again v_1 is determined by the rest of the data). We write

$$\gamma : \widetilde{\mathcal{R}}_{1+2} \to \overline{\mathcal{R}}_{1+2}$$

for the projection. It is ind-pro-smooth. Finally, we have a natural map

$$\begin{array}{rcl} \delta : \overline{\mathcal{R}}_{1+2} & \to & \mathcal{R}^{12}_{12} = \{((x_1, x_2), \mathcal{E}, f, v)\}/\sim \\ ((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, g_1^{-1} f_2, v_1, v_2) & \mapsto & ((x_1, x_2), \mathcal{E}_2, f_1 g_1^{-1} f_2, v_2). \end{array}$$

Note that δ factors as $\delta = \delta' \delta''$ where $\delta'' : \overline{\mathcal{R}}_{1+2} \to \bullet$ is an ind-closed embedding of finite codimension and $\delta' : \bullet \to \mathcal{R}^{12}_{12}$ is defined by the Cartesian square

$$\begin{array}{ccc} \bullet & \stackrel{\delta'}{\to} & \mathcal{R}^{12}_{12} \\ \downarrow & & \downarrow \\ G^1_{12} \frac{\times_{X_1 \times X_2}}{G_{12}} G^2_{12}/G_{12} & \stackrel{d}{\to} & G^{12}_{12}/G_{12} \end{array}$$

where the bottom row is simply the top row for $\mathbf{N} = 0$, and the vertical maps forget v_1, v_2, v. It is well-known that d is ind-projective; this fact shows up already in [4] and essentially follows from Lemma 1.3. It follows that δ is also ind-projective, meaning that in each piece of the inductive structure, δ is Zariski-locally of the form

$$Y \times \mathbb{A} \xrightarrow{f \times id} Z \times \mathbb{A}$$

for $f : Y \to Z$ a projective map between schemes of finite type, and $\mathbb{A}$ some affine space of countable dimension. In fact, δ is an isomorphism over U, while over the diagonal its fibers are products of closed subvarieties of affine Grassmannians. Furthermore, δ is G_{12}-equivariant.

2.4. The global convolution diagram is

$$\mathcal{R}^1_1 \times \mathcal{R}^2_2 \xleftarrow{\alpha} \mathcal{R}_{1+2} \xleftarrow{\beta} \widetilde{\mathcal{R}}_{1+2} \xrightarrow{\gamma} \overline{\mathcal{R}}_{1+2} \xrightarrow{\delta} \mathcal{R}^{12}_{12}\,.$$

As we have explained, α, β are $G_{12} \times_{X_1 \times X_2} G_{12}$-equivariant, γ is the quotient map by the free action of the right-hand G_{12}, and δ is equivariant for the remaining copy of G_{12}. We have already explained how α defines a map

$$\alpha^* : H^{BM, G_1 \times G_2}_{*-2\dim \mathbf{N}_1 \times \mathbf{N}_2}(\mathcal{R}^1_1 \times \mathcal{R}^2_2) \to H^{BM, G_{12} \times_{X_1 \times X_2} G_{12}}_{*-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}(\mathcal{R}_{1+2}).$$

Everything else works out essentially as in [1], as we now indicate. First, recall the $G_{12} \times_{X_1 \times X_2} G_{12}$-equivariant Cartesian diagram 2.2:

$$\begin{array}{ccc} \widetilde{\mathcal{R}}_{1+2} & \xrightarrow{\beta} & \mathcal{R}_{1+2} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{T}}_{1+2} & \xrightarrow{b} & \mathcal{T}^1_{12} \times_{X_1 \times X_2} \mathcal{R}^2_{12} \end{array}$$

and recall that $\widetilde{\mathcal{T}}_{1+2}$ is nothing other than $G^1_{12} \times_{X_1 \times X_2} \mathcal{R}^2_{12}$. Thus we may write

$$\begin{array}{rcl} b & = & pr_1 b \times_{X_1 \times X_2} pr_2 b \\ pr_1 b & = & \psi\phi \\ pr_2 b & = & pr_2 \end{array}$$

where we have factored $pr_1 b$ as

$$G^1_{12} \times_{X_1 \times X_2} \mathcal{R}^2_{12} \xrightarrow{\phi} G^1_{12} \times_{X_1 \times X_2} \mathbf{N}_{12} \xrightarrow{\psi} \mathcal{T}^1_{12}\,.$$

It follows that

$$b^* \omega_{\mathcal{T}^1_{12} \times_{X_1 \times X_2} \mathcal{R}^2_{12}}[-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}] \cong \omega_{\widetilde{\mathcal{T}}_{1+2}}[-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}]$$

and hence by base change we have have a map

$$\beta^* \omega_{\mathcal{R}_{1+2}}[-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}] \to \omega_{\widetilde{\mathcal{R}}_{1+2}}[-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}]. \tag{2.3}$$

This map is equivariant, and it therefore determines a 'pullback with support' map:

$$\beta^* : H^{BM, G_{12} \times_{X_1 \times X_2} G_{12}}_{*-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}(\mathcal{R}_{1+2}) \to H^{BM, G_{12} \times_{X_1 \times X_2} G_{12}}_{*-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}}(\widetilde{\mathcal{R}}_{1+2}).$$

Since it is a G_{12}-torsor, γ induces an isomorphism

$$\gamma^* : H^{BM, G_{12}}_{*-2\dim \mathbf{N}_{12}}(\overline{\mathcal{R}}_{1+2}) \xrightarrow{\sim} H^{BM, G_{12} \times_{X_1 \times X_2} G_{12}}_{*-2\dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}}(\widetilde{\mathcal{R}}_{1+2}).$$

Finally since it is ind-proper and equivariant, δ induces a map

$$\delta_* : H^{BM, G_{12}}_{*-2\dim \mathbf{N}_{12}}(\overline{\mathcal{R}}_{1+2}) \to H^{BM, G_{12}}_{*-2\dim \mathbf{N}_{12}}(\mathcal{R}^{12}_{12}).$$

2.5. Recall that (dual) specialization maps commute with pullbacks along smooth maps and pushforwards along proper maps, and are compatible with equivariance with respect to smooth group schemes. Therefore, since every space in sight is a reasonably nice ind-scheme and the groups G_S are pro-smooth over X^S, we have (dual) specialization maps to the diagonal $X_0 \subset X_1 \times X_2$:

$$\begin{array}{lccc}
s_1: & H^{BM,G_1\times G_2}_{*-2\dim \mathbf{N}_1\times\mathbf{N}_2}(\mathcal{R}^1_1\times\mathcal{R}^2_2) & \to & H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}\mathbf{N}_0}(\mathcal{R}^0_0\times_{X_0}\mathcal{R}^0_0) \\
s_2: & H^{BM,G_{12}\times_{X_1\times X_2}G_{12}}_{*-2\dim \mathbf{N}_1\times\mathbf{N}_2}(\mathcal{R}^1_1\times\mathcal{R}^2_2) & \to & H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}\mathbf{N}_0}(\mathcal{R}^0_0\times_{X}\mathcal{R}^0_0) \\
s_3: & H^{BM,G_{12}\times_{X_1\times X_2}G_{12}}_{*-2\dim \mathbf{N}_{12}\times_{X_1\times X_2}\mathbf{N}_{12}}(\mathcal{R}_{1+2}) & \to & H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}\mathbf{N}_0}(\mathcal{R}^0_0\times_{X_0}\mathcal{R}^0_0) \\
s_4: & H^{BM,G_{12}\times_{X_1\times X_2}G_{12}}_{*-2\dim \mathbf{N}_{12}\times_{X_1\times X_2}G_{12}}(\widetilde{\mathcal{R}}_{1+2}) & \to & H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}G_0}(\widetilde{\mathcal{R}}_0) \\
s_5: & H^{BM,G_{12}}_{*-2\dim \mathbf{N}_{12}}(\overline{\mathcal{R}}_{1+2}) & \to & H^{BM,G_0}_{*-2\dim \mathbf{N}_0}(\overline{\mathcal{R}}_0) \\
s_6: & H^{BM,G_{12}}_{*-2\dim \mathbf{N}_{12}}(\mathcal{R}^{12}_{12}) & \to & H^{BM,G_0}_{*-2\dim \mathbf{N}_0}(\mathcal{R}^0_0).
\end{array}$$

Here $\widetilde{\mathcal{R}}_0$, $\overline{\mathcal{R}}_0$ are respectively locally trivial $p^{-1}(\mathcal{R}\times\mathcal{R})$, $q(p^{-1}(\mathcal{R}\times\mathcal{R}))$-bundles over X_0 in the notations of diagram (3.2) in the main paper. In fact, the restriction of the convolution diagram to X_0 induces the following maps between the targets of the specialization maps:

$$\begin{array}{lcc}
H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}\mathbf{N}_0}(\mathcal{R}^0_0\times_{X_0}\mathcal{R}^0_0) & \xrightarrow{id} & H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}\mathbf{N}_0}(\mathcal{R}^0_0\times_{X_0}\mathcal{R}^0_0) \\
& \xrightarrow{id} & H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}\mathbf{N}_0}(\mathcal{R}^0_0\times_{X_0}\mathcal{R}^0_0) \\
& \xrightarrow{\beta_0^*} & H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}G_0}(\widetilde{\mathcal{R}}_0) \\
& \xrightarrow{(\gamma_0^*)^{-1}} & H^{BM,G_0\times_{X_0}G_0}_{*-2\dim \mathbf{N}_0\times_{X_0}G_0}(\widetilde{\mathcal{R}}_0) \\
& \xrightarrow{(\delta_0)_*} & H^{BM,G_0}_{*-2\dim \mathbf{N}_0}(\mathcal{R}^0_0)
\end{array}$$

I claim that the maps $\alpha^*, \beta^*, (\gamma^*)^{-1}, \delta_*$ are intertwined with $id, \beta_0^*, (\gamma_0^*)^{-1}, (\delta_0)_*$ by the (dual) specialization maps. For α^*, $(\gamma^*)^{-1}$ it is a consequence of ind-pro-smoothness of α, γ (and also pro-smoothness of G_{12}). For δ_* it is a consequence of ind-properness. For β^*, it is because the map

$$(\beta_0)^*\omega_{\mathcal{R}^0_0\times_{X_0}\mathcal{R}^0_0}[-2\dim \mathbf{N}_0\times_{X_0}\mathbf{N}_0] \to \omega_{\widetilde{\mathcal{R}}_0}[-2\dim \mathbf{N}_0\times_{X_0}G_0]$$

defined using the Cartesian square:

$$\begin{array}{ccc}
\widetilde{\mathcal{R}}_0 & \xrightarrow{\beta_0} & \mathcal{R}^0_0 \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{T}}^0_0 & \xrightarrow{b_0} & \mathcal{T}^0_0\times_{X_0}\mathcal{R}^0_0
\end{array}$$

obtained by restricting diagram 2.2 to X_0, factors as:

$$\begin{array}{lcc}
(\beta_0)^*\omega_{\mathcal{R}^0_0\times_{X_0}\mathcal{R}^0_0}[-2\dim \mathbf{N}_0\times_{X_0}\mathbf{N}_0] & \cong & (\beta_0)^* i_1^!\omega_{\mathcal{R}_{1+2}}[-2\dim \mathbf{N}_{12}\times_{X_1\times X_2}\mathbf{N}_{12}+2] \\
& \xrightarrow{can} & i_2^!\beta^*\omega_{\mathcal{R}_{1+2}}[-2\dim \mathbf{N}_{12}\times_{X_1\times X_2}\mathbf{N}_{12}+2] \\
& \xrightarrow{i_2^{!}[2]\,(2.3)} & i^!\omega_{\widetilde{\mathcal{R}}_{1+2}}[-2\dim \mathbf{N}_{12}\times_{X_1\times X_2}G_{12}+2] \\
& \cong & \omega_{\widetilde{\mathcal{R}}_0}[-2\dim \mathbf{N}_0\times_{X_0}G_0].
\end{array}$$

Here *can* is the canonical map arising from the base change isomorphism, (2.3) denotes the map of equation 2.3, and i_1, i_2 denote the appropriate inclusions of the diagonal subspaces. The consequence is the following formula:

$$s_6\delta_*(\gamma^*)^{-1}\beta^*\alpha^* = (\delta_0)_*(\gamma_0^*)^{-1}\beta_0^* s_1 : H^{BM,G_1\times G_2}_{*-2\dim \mathbf{N}_1\times\mathbf{N}_2}(\mathcal{R}^1_1\times\mathcal{R}^2_2) \to H^{BM,G_0}_{*-2\dim \mathbf{N}_0}(\mathcal{R}^0_0).$$

2.6. Now each (dual) specialization map s_n factors as $s'_n j_n^*$ where j_n^* is the restriction map to the equivariant Borel-Moore homology of the part lying over U, and s'_n is some other map. Furthermore, the restriction of the convolution diagram to U induces the following maps between the targets of the restriction maps:

$$\begin{aligned}
& H^{BM,(G_1\times G_2)|_U}_{*-2\dim(\mathbf{N}_1\times\mathbf{N}_2)|_U}((\mathcal{R}_1^1\times\mathcal{R}_2^2)|_U) \\
\to\ & H^{BM,(G_1\times G_2)|_U\times_U(G_1\times G_2)|_U}_{*-2\dim(\mathbf{N}_1\times\mathbf{N}_2)|_U}((\mathcal{R}_1^1\times\mathcal{R}_2^2)|_U) \\
\to\ & H^{BM,(G_1\times G_2)|_U\times_U(G_1\times G_2)|_U}_{*-2\dim(\mathbf{N}_1\times\mathbf{N}_2)|_U\times_U(\mathbf{N}_1\times\mathbf{N}_2)|_U}((\mathcal{R}_1^1\times\mathbf{N}_2)|_U\times_U(\mathbf{N}_1\times\mathcal{R}_2^2)|_U) \\
\to\ & H^{BM,((G_1\times_{X_1}G_1)\times(G_2\times_{X_2}G_2))|_U}_{*-2\dim((\mathbf{N}_1\times_{X_1}G_1)\times(G_2\times_{X_2}\mathbf{N}_2))|_U}((\widetilde{\mathcal{R}}_1\times(G_2\times_{X_2}\mathcal{R}_2^2))|_U) \\
\to\ & H^{BM,(G_1\times G_2)|_U}_{*-2\dim(\mathbf{N}_1\times\mathbf{N}_2)|_U}((\mathcal{R}_1^1\times\mathcal{R}_2^2)|_U) \\
\to\ & H^{BM,(G_1\times G_2)|_U}_{*-2\dim(\mathbf{N}_1\times\mathbf{N}_2)|_U}((\mathcal{R}_1^1\times\mathcal{R}_2^2)|_U)
\end{aligned}$$

Let us explain what each map does:

(1) The first map views any $(G_1\times G_2)|_U$-equivariant class as also equivariant for the trivial actions of the left-hand copy of G_2, and the right-hand copy of G_1, in $(G_1\times G_2)|_U\times_U(G_1\times G_2)|_U$.

(2) The second map pulls this back along the $(\mathbf{N}_2\times\mathbf{N}_1)|_U$-bundle map (i.e. multiplies fiberwise by the equivariant fundamental class of $\mathbf{N}(\mathcal{O})\times\mathbf{N}(\mathcal{O})$).

(3) The third map starts by rewriting $(\mathcal{R}_1^1\times\mathbf{N}_2)|_U\times_U(\mathbf{N}_1\times\mathcal{R}_2^2)|_U$ as $((\mathcal{R}_1^1\times_{X_1}\mathbf{N}_1)\times(\mathbf{N}_2\times_{X_2}\mathcal{R}_2^2))|_U$, and rewriting the action of $(G_1\times G_2)|_U\times_U(G_1\times G_2)|_U$ as one of $((G_1\times_{X_1}G_1)\times(G_2\times_{X_2}G_2))|_U$. By definition, $\widetilde{\mathcal{R}}_1$ is the locally trivial $p^{-1}(\mathcal{R}\times\mathcal{R})$-bundle on X_1 given as

$$\widetilde{\mathcal{R}}_1=\mathbf{N}_1\times_{\mathbf{N}_1^1}(G_1^1\times_{X_1}\mathbf{N}_1).$$

The $G_1\times_{X_1}G_1$-equivariant map from here to $\mathcal{R}_1^1\times_{X_1}\mathbf{N}_1$ is given as the product (over X_1) of the quotient by the right-hand copy of G_1 with the projection to the right-hand copy of $\mathbf{N}_1$. The 'pullback with support' map

$$H^{BM,G_1\times_{X_1}G_1}_{*-2\dim\mathbf{N}_1\times_{X_1}\mathbf{N}_1}(\mathcal{R}_1^1\times_{X_1}\mathbf{N}_1)\to H^{BM,G_1\times_{X_1}G_1}_{*-2\dim\mathbf{N}_1\times_{X_1}G_1}(\widetilde{\mathcal{R}}_1)$$

corresponds to the composition of usual 'pullback with support' (spread out over X_1) with multiplication by $H^*_{G_1}(X_1)$ under the identification

$$H^{BM,G_1\times_{X_1}G_1}_{*-2\dim\mathbf{N}_1\times_{X_1}\mathbf{N}_1}(\mathcal{R}_1^1\times_{X_1}\mathbf{N}_1)=H^{BM,G_1}_{*-2\dim\mathbf{N}_1}(\mathcal{R}_1^1)\otimes_{H^*(X_1)}H^*_{G_1}(X_1).$$

Meanwhile, the 'pullback with support' (actually, here no support is required) map

$$H^{BM,G_2\times_{X_2}G_2}_{*-2\dim\mathbf{N}_2\times_{X_2}\mathbf{N}_2}(\mathbf{N}_2\times_{X_2}\mathcal{R}_2^2)\to H^{BM,G_2\times_{X_2}G_2}_{*-2\dim G_2\times_{X_2}\mathbf{N}_2}(G_2\times_{X_2}\mathcal{R}_2^2)$$

is isomorphic simply to the multiplication map

$$H^*_{G_2}(X_2)\otimes_{H^*(X_2)}H^{BM,G_2}_{*-2\dim\mathbf{N}_2}(\mathcal{R}_2^2)\to H^{BM,G_2}_{*-2\dim\mathbf{N}_2}(\mathcal{R}_2^2).$$

(4) The fourth map is the isomorphism, and the fifth is the identity.

The result is that the composition of all these maps is the identity. On the other hand, since the restriction maps j_n^* intertwine these maps with the corresponding maps on the $X_1 \times X_2$ level, we have the following:

$$\begin{aligned}(\delta_0)_*(\gamma_0^*)^{-1}\beta_0^* s_1 &= s_6 \delta_*(\gamma^*)^{-1}\beta^*\alpha^* \\ &= s_6' j_6^* \delta_*(\gamma^*)^{-1}\beta^*\alpha^* \\ &= s_6' j_1^*.\end{aligned}$$

2.7. Finally, note that this last map $s_6' j_1^*$ is symmetric with respect to the automorphism τ of $H^{BM,G_1\times G_2}_{*-2\dim \mathbf{N}_1\times\mathbf{N}_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2)$ induced by the degree 2 automorphisms of $G_1 \times G_2$, $\mathcal{R}_1^1 \times \mathcal{R}_2^2$ which switch the factors (and also exchange 1 with 2). Therefore, $(\delta_0)_*(\gamma_0^*)^{-1}\beta_0^* s_1$ has the same property. But, taking $X = \mathbb{C}$, we identify the domain

$$H^{BM,G_1\times G_2}_{*-2\dim \mathbf{N}_1\times\mathbf{N}_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) = H^{BM,G(\mathcal{O})}_{*-2\dim \mathbf{N}(\mathcal{O})}(\mathcal{R}) \otimes H^{BM,G(\mathcal{O})}_{*-2\dim \mathbf{N}(\mathcal{O})}(\mathcal{R})$$

and the target

$$H^{BM,G_0}_{*-2\dim \mathbf{N}_0}(\mathcal{R}_0^0) = H^{BM,G(\mathcal{O})}_{*-2\dim \mathbf{N}(\mathcal{O})}(\mathcal{R}).$$

The map $(\delta_0)_*(\gamma_0^*)^{-1}\beta_0^* s_1$ is the usual convolution (s_1 is an isomorphism) while τ is the standard twist. Therefore, the Coulomb branch is commutative as claimed.

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