NOTES FOR BEILINSON-DRINFELD SEMINAR - EVEN SHORTER VERSION

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Abstract. We use Steenrod's construction to prove that the quantum Coulomb branch is a Frobenius-constant quantization.

Contents

1. Overview

This talk is essentially about 'power operations'. In the first part I will introduce some power operations in homological algebra and in quantization theory. In the second part I will explain how these are related in the context of 'quantum Coulomb branch' of Braverman-Finkelberg-Nakajima.

2. Steenrod's construction

Notation 2.1.

- G - alg. gp./$\mathbb{C}$
- X - alg. var./$\mathbb{C}$, $G \curvearrowright X$
- p - odd prime
- $D_G(X)$ - G-equivariant bounded constructible derived category of sheaves of $\mathbb{F}_{\mathrm{p}}$-modules on X^{an}
- Σ - suspension functor
- $\mathbb{F}_{\mathrm{p}}$ - equivariant constant sheaf
- ω - equivariant dualizing complex
- μ_{p} - subgroup of $\mathbb{C}^*$ of order p
- $X^p = Map(\mu_{\mathrm{p}}, X) \circlearrowleft G^p \rtimes \mu_{\mathrm{p}}$
- $\wedge \boxtimes p : D_G(X) \to D_{G^p}(X^p)$ functor of p^{th} external tensor power
- Biadjoint functors $Res : D_{G^p \rtimes \mu_{\mathrm{p}}}(X^p) \rightleftarrows D_{G^p}(X^p) : Ind$

Theorem-Definition 2.1 (Steenrod)**.** *There is a functor $St : D_G(X) \to D_{G^p \rtimes \mu_p}(X^p)$ satisfying $Res \circ St \cong \wedge \boxtimes p$. In a diagram:*

$$\begin{array}{ccc} D_G(X) & \xrightarrow{\wedge \boxtimes p} & D_{G^p}(X^p) \\ \| & & \uparrow \mathrm{Res} \\ D_G(X) & \xrightarrow{St} & D_{G^p \rtimes \mu_p}(X^p). \end{array}$$

commutes.

Facts 2.2.

(1) $St\Sigma \cong \Sigma^p St$.
(2) Given $f, g : A \to B$ in $D_G(X)$, can find a morphism $h : A^{\boxtimes p} \to B^{\boxtimes p}$ in $D_{G^p \rtimes \mu}(X^p)$ s.t. $St(f+g) - St(f) - St(g)$ equals composition

$$Av(h) : St(A) \xrightarrow{adjunction} \mathrm{Ind}(A^{\boxtimes p}) \xrightarrow{\mathrm{Ind}(h)} \mathrm{Ind}(B^{\boxtimes p}) \xrightarrow{adjunction} St(B).$$

(3) St commutes with multiplication by $\mathbb{F}_p$.
(4) $St(\mathbb{F}_p) \cong \mathbb{F}_p$.
(5) $St(\omega) \cong \omega$.
(6) St is monoidal.

Consequence 2.3.

(1) Identify G-equivariant $\mathbb{F}_p$-cohomology $H^n_G(X)$ with $\mathrm{Hom}_{D_G(X)}(\mathbb{F}_p, \Sigma^n \mathbb{F}_p)$. So St induces non-linear maps

$$St : H^n_G(X) \to H^{pn}_{G^p \rtimes \mu_p}(X^p).$$

Pull back along diagonal Δ:

$$\Delta^* \circ St : H^n_G(X) \to H^{pn}_{G \times \mu_p}(X) \xrightarrow{Kunneth\cong} (H^*_G(X) \otimes H^*_{\mu_p}(*))^{pn} = (H^*_G(X)[a, \hbar])^{pn}.$$

Here we used: $H^*_{\mu_p}(*) = \mathbb{F}_p[a, \hbar]$, $\deg(a) = 1, \deg(\hbar) = 2$. These are *linear*, because Δ^* commutes with Av and $Av : H^*_G(X) \to H^*_{G \times \mu_p}(X)$ equals 0. Thus have a map of graded vector spaces

$$St' : H^*_G(X)^{(1)} \to H^*_G(X)[a, \hbar]$$

Here $H^*_G(X)^{(1)}$ is the *Frobenius twist* of $H^*_G(X)$. It is the algebra obtained from $H^*_G(X)$ by multiplying degrees by p and introducing signs:

$$x^{(1)} y^{(1)} = (-1)^{\deg(x) \deg(y) \binom{p}{2}} (xy)^{(1)}.$$

Fact: St' is a map of algebras. Coefficients of monomials $\hbar^m, a\hbar^m$ are the *Steenrod operations*, up to some correction factors.

(2) Equivariant Borel-Moore homology of X is defined as

$$H^{BM,G}_n := Hom_{D_G(X)}(\mathbb{F}_p, \Sigma^n \omega)$$

and we have maps of sets

$$St : H^{BM,G}_n(X) \to H^{BM,G^p \rtimes \mu_p}_{pn}(X^p).$$

These are not linear, and unlike for cohomology there is no way in general to get something linear out of them. But we will see later that we can do something like that if X has a suitably symmetric multiplication.

Fact 2.4. The graded vector space $H_*^{BM,G}(X)$ is a module over $H_G^*(X)$, and in particular over $H_G^*(*)$. The family of maps

$$St : H_n^{BM,G}(X) \to H_{pn}^{BM,G^p \rtimes \mu_p}(X^p)$$

is compatible with the multiplication by

$$St : H_G^n(X) \to H_{G^p \rtimes \mu_p}^{pn}(X^p)$$

in the natural way.

Example 2.5. Take $X = *, G = \mathbb{C}^*$. We have $H_{\mathbb{C}^*}^* = \mathbb{F}_p[c]$ with $\deg(c) = 2$. The map of algebras

$$St' : \mathbb{F}_p[c]^{(1)} \to \mathbb{F}_p[c, a, \hbar]$$

factors through the inclusion of $\mathbb{F}_p[c, \hbar]$ for degree reasons, and sends c to $c^p - \hbar^{p-1}c$. Identifying $\mathbb{F}_p[c]$ with $\mathcal{O}(\mathfrak{g}_m)$ and taking Spec we get a $\mathbb{G}_m$-equivariant map

$$AS_\hbar : \mathfrak{g}_m \times \mathbb{G}_a \to \mathfrak{g}_m^{(1)}.$$

AS_0 is the Frobenius map, while AS_1 is the Artin-Schreier map. This observation generalizes in the natural way to $X = *, G = T$ complex torus.

3. Frobenius-constant quantizations

$AS_\hbar$ plays a central role in the following theory. Let A be a commutative $\mathbb{F}_p$-algebra.

Definition 3.1.
(1) A *quantization* of A is a flat (i.e. torsion-free) associative $\mathbb{F}_p[\hbar]$-algebra $A_\hbar$ satisfying $A_\hbar/\hbar = A$.
(2) A *Frobenius-constant quantization* of A is a quantization $A_\hbar$ of A together with a map of algebras

$$F_\hbar : A^{(1)} \to Z(A_\hbar)$$

such that $F_\hbar \mod \hbar$ is the Frobenius map. Here $Z(A_\hbar)$ is the center of $A_\hbar$. The Frobenius twist $A^{(1)}$ has meaning if we are in an equivariant setting, see example.

Example 3.2. Consider $\mathbb{G}_m = \operatorname{Spec} \mathbb{F}_p[x, x^{-1}]$. Set $A = \mathcal{O}(T^* \mathbb{G}_m) = \mathbb{F}_p[x, x^{-1}, y]$. Its canonical quantization is the Weyl algebra

$$A_\hbar = \mathcal{D}_\hbar(\mathbb{G}_m) = \mathbb{F}_p[\hbar]\langle x, x^{-1}, \partial\rangle/([\partial, x] = \hbar).$$

These are both $\mathbb{G}_m \times \mathbb{G}_m$-equivariant (regular action and homotheties) with x in bidegree $(1, 0)$ and y, ∂ in bidegree $(-1, 2)$. Note that $x^p, \partial^p \in Z(A_\hbar)$. So we can set

$$\begin{array}{rccc} F_\hbar : & A^{(1)} & \to & Z(A_\hbar) \\ & x_i & \mapsto & x_i^p \\ & y_i & \mapsto & \partial_i^p. \end{array}$$

$F_\hbar$ is $\mathbb{G}_m \times \mathbb{G}_m$-equivariant. Taking invariants for the regular action we get

$$\begin{array}{ccc} \mathbb{F}_p[xy] & \to & \mathbb{F}_p[\hbar, x\partial] \\ xy & \mapsto & x^p \partial^p. \end{array}$$

It is an exercise to show that $x^n \partial^n = \prod_{i=0}^{n-1}(x\partial - i\hbar)$. Thus we have

$$x^p \partial^p = \prod_{i=0}^{p-1}(x\partial - i\hbar) = (x\partial)^p - \hbar^{p-1} x\partial.$$

Therefore if we identify $\mathbb{F}_p[xy]$ with $\mathbb{F}_p[c]$ and $\mathbb{F}_p[x\partial, \hbar]$ with $\mathbb{F}_p[c, \hbar]$ we recover $AS_\hbar$. Likewise if we replace $\mathbb{G}_m$ with the $\mathbb{F}_p$-split Langlands dual torus $T^\vee$ to complex torus T.

4. End of first part - time for a break

5. Coulomb branch

5.1. $Gr, \mathcal{T}, \mathcal{R}$.

- G - complex reductive algebraic group
- T - maximal torus of G
- $\mathbf{N}$ - G-module of dimension $d < \infty$
- $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$.

For simplicity assume that p is large. We consider three 'reasonable' ind-schemes of interest.

(1) The *affine Grassmannian* Gr is an ind-projective ind-scheme whose $\mathbb{C}$-points are $G(\mathcal{K})/G(\mathcal{O})$. Informally,
$$Gr = G(\mathcal{K})/G(\mathcal{O}).$$
Its $G(\mathcal{O})$-orbits are parameterized by dominant cocharacters λ of T. Corresponding orbit closures, Gr^λ, exhaust Gr. Gr parameterizes pairs
$$(\mathcal{E}, f)$$
of a principal G-bundle $\mathcal{E}$ on the formal disc Δ and a trivialization f of $\mathcal{E}$ on punctured formal disc Δ^o, up to isomorphism.

(2) $\mathbf{N}(\mathcal{O})$ is a $G(\mathcal{O})$-module, and we consider the associated $\mathbf{N}(\mathcal{O})$-bundle
$$\mathcal{T} := G(\mathcal{K}) \overset{\times}{_{G(\mathcal{O})}} \mathbf{N}(\mathcal{O}).$$
It is an infinite-dimensional vector bundle over Gr. Its ind-scheme structure is given by pieces $\mathcal{T}^\lambda = \mathcal{T}|_{Gr^\lambda}$. We write $\mathcal{T}^\lambda_n$ for finite-dimensional quotient bundle $\mathcal{T}^\lambda / t^n.\mathcal{T}^\lambda$. The rank of this bundle is $\dim_{Gr^\lambda}(\mathcal{T}^\lambda_n) = nd$. Moduli: $\mathcal{T}$ parameterizes triples up to isomorphism
$$(\mathcal{E}, f, v)$$
where $\mathcal{E}, f$ are as in Gr and v is an $\mathbf{N}$-section of $\mathcal{E}$.

(3) We have $\mathcal{T} \to \mathbf{N}(\mathcal{K})$ by multiplication. $\mathcal{R}$ is defined to be the preimage of $\mathbf{N}(\mathcal{O})$. It is a closed sub-ind-scheme of $\mathcal{T}$, given by pieces $\mathcal{R}^\lambda = \mathcal{R}|_{Gr^\lambda} \subset \mathcal{T}^\lambda$. The fibers over Gr are vector subspaces of $\mathbf{N}(\mathcal{O})$ of finite codimension. Each $\mathcal{R}^\lambda$ contains the bundle $t^n.\mathcal{T}^\lambda$ for some $n >> 0$, and we will consider the fiberwise quotient $\mathcal{R}^\lambda_n = \mathcal{R}^\lambda / t^n.\mathcal{T}^\lambda$. (Draw the picture). The possible n go to ∞ as $\lambda \to \infty$. $\mathcal{R}$ parameterizes triples up to isomorpihsm
$$(\mathcal{E}, f, v)$$
as in $\mathcal{T}$ with the additional condition that $f(v)$ extends over Δ.

Each of $Gr, \mathcal{T}, \mathcal{R}$ and approximations $Gr^\lambda, \mathcal{T}^\lambda_n, \mathcal{R}^\lambda_n$ have natural actions of $G(\mathcal{O}) \rtimes \mathbb{C}^*$. In the moduli interpretation, $G(\mathcal{O})$ acts simply by changing the trivialization f, while $\mathbb{C}^*$ acts by automorphisms of Δ. $z.(\mathcal{E}, f, v) = (z_* \mathcal{E}, z_* f, z_* v)$.

Example 5.1. $G = \mathbb{C}^*$, $\mathbf{N} = \mathbb{C}_{-r}$, $r \geqslant 0$. Then on $\mathbb{C}$-points we identify:

(1) $Gr = \mathbb{Z}$

(2) $\mathcal{T} = \mathbb{Z} \times \mathbb{C}_{-r}[[t]]$
(3) $\mathcal{R} = \mathbb{Z}_{\leqslant 0} \times \mathbb{C}_{-r}[[t]] \cup \{1\} \times t^r \mathbb{C}_{-r}[[t]] \cup \{2\} \times t^{2r} \mathbb{C}_{-r}[[t]] \cup \ldots$.

5.2. **Borel-Moore homology.** Fix λ, n such that $\mathcal{R}_n^\lambda$ exists. Let $\mathcal{G}$ be one of the groups $G(\mathcal{O}), G(\mathcal{O}) \rtimes \mathbb{C}^*$. Since $\mathcal{R}_{n+1}^\lambda \to \mathcal{R}_n^\lambda$ is an $\mathbf{N}$-bundle, the pullback induces isomorphisms

$$H_i^{BM,\mathcal{G}}(\mathcal{R}_n^\lambda) \cong H_{i-2d}^{BM,\mathcal{G}}(\mathcal{R}_{n+1}^\lambda).$$

Therefore the *shifted Borel-Moore homology*

$$H_{i-2nd}^{BM,\mathcal{G}}(\mathcal{R}_n^\lambda) = H_{i-2\dim_{Gr^\lambda}(\mathcal{T}_n^\lambda)}^{BM,\mathcal{G}}(\mathcal{R}_n^\lambda)$$

is independent of n. We can therefore formally drop the 'n' and write it as

$$H_{i-2\dim_{Gr^\lambda}(\mathcal{T}^\lambda)}^{BM,\mathcal{G}}(\mathcal{R}^\lambda).$$

We can then take a colimit as $\lambda \to \infty$ (pushing forward along closed embeddings) to obtain the formally shifted Borel-Moore homology:

$$H_{i-2\dim_{Gr}(\mathcal{T})}^{BM,\mathcal{G}}(\mathcal{R}).$$

Definition 5.2. (1) The *Coulomb branch* is the graded $H_G^*(*)$-module

$$A = H_{*-2\dim_{Gr}(\mathcal{T})}^{BM,G(\mathcal{O})}(\mathcal{R}).$$

(2) The *quantum Coulomb branch* is the graded $H_{G\times\mathbb{C}^*}^*(*)$-module

$$A_\hbar = H_{*-2\dim_{Gr}(\mathcal{T})}^{BM,G(\mathcal{O})\rtimes\mathbb{C}^*}(\mathcal{R}).$$

$A_\hbar$ is a flat graded deformation of A over $\mathbb{F}_{\mathrm{p}}[\hbar] = H_{\mathbb{C}^*}^*(*)$.

Fact 5.3. (1) A, $A_\hbar$ are evenly graded.
(2) A, $A_\hbar$ have algebra structures via '*convolution*'.
(3) A is commutative, and $A_\hbar$ is its quantization. If I have time at the end I will explain how this works.

Remark 5.4. We can replace $G(\mathcal{O})$-equivariance by G-equivariance in definition of A, $A_\hbar$. But to define the convolution we need to use $G(\mathcal{O})$-equivariance.

Example 5.5. (1) $G = \mathbb{C}^*$, $\mathbf{N} = 0$. Then $A_\hbar$ is the Weyl algebra $\mathbb{F}_{\mathrm{p}}[\hbar]\langle x^\pm, \partial\rangle/([\partial, x] = \hbar)$. Equivariant BM homology of point $n \in \mathbb{Z}$ is identified with $\mathbb{F}_{\mathrm{p}}[\hbar, x\partial].x^n$.
(2) $G = \mathbb{C}^*$, $\mathbf{N} = \mathbb{C}_{-r}$, $r \geqslant 0$. Then $A_\hbar$ is the subalgebra of the Weyl algebra with basis

$$\ldots x^{-2}, x^{-1}, 1, \prod_{i=1}^{r}(rx\partial - i\hbar)x, \prod_{i=1}^{2r}(rx\partial - i\hbar)x^2, \ldots$$

over $\mathbb{F}_{\mathrm{p}}[\hbar, x\partial]$.

5.3. **Frobenius-constancy.**

Main Theorem 5.1. *$A_\hbar$ is a Frobenius-constant quantization.*

I will construct the map $F_\hbar : A^{(1)} \to A_\hbar$. There is an ind-scheme $\mathcal{R}_{(p)}$ over $\mathbb{G}_a$ parameterizing:

$$\left\{ (x, \mathcal{E}, f, v) : \begin{array}{ll} x & \in \mathbb{C} \\ \mathcal{E} & \text{a principal } G\text{-bundle over } \Delta_{\pi^{-1}x} \\ f & \text{a trivialization of } \mathcal{E} \text{ over } \Delta^o_{\pi^{-1}x} \\ v & \text{an } \mathbf{N}\text{-section of } \mathcal{E} \text{ such that } f(v) \text{ extends over } \Delta_{\pi^{-1}x} \end{array} \right\} / \sim .$$

Here by definition:

- $\pi : \mathbb{G}_a \to \mathbb{G}_a$ is the map $x \mapsto x^p$.
- $\Delta_{\pi^{-1}x}$ is the formal neighborhood in $\mathbb{G}_a$ of $\pi^{-1}x$. In a formula,
$$\Delta_{\pi^{-1}x} = \mathrm{Spec}(\varprojlim_n \mathbb{C}[t]/(t^p - x)^n).$$
- $\Delta^o_{\pi^{-1}x}$ is the punctured formal neighborhood,
$$\Delta^o_{\pi^{-1}x} = \mathrm{Spec}((\varprojlim_n \mathbb{C}[t]/(t^p - x)^n)[(t^p - x)^{-1}]).$$

There is a global version of $G(\mathcal{O})$ acting on $\mathcal{R}_{(p)}$, but for simplicity we just consider the actions of G by

$$g.(x, \mathcal{E}, f, v) = (x, \mathcal{E}, g \circ f, v).$$

and of $\mathbb{C}^*$ by

$$z.(x, \mathcal{E}, f, v) = (z^p x, z_* \mathcal{E}, z_* f, z_* v).$$

Also $\mathcal{R}_{(p)}$ is embedded in $\mathcal{T}_{(p)}$, defined the same way as $\mathcal{R}_{(p)}$ but without the condition on v.

$$\mathcal{R}_{(p)} \subset \mathcal{T}_{(p)}\,.$$

The version of $\mathcal{R}$ with $\mathbf{N} = 0$ is called $Gr_{(p)}$. $\mathcal{T}_{(p)}$ is an infinite-dimensional vector bundle over $Gr_{(p)}$, with a fiberwise shift map $(t^p - x)$. We may define

$$H^{BM,\mathcal{G}}_{*-2\dim_{Gr_{(p)}} \mathcal{T}^{(p)}}(\mathcal{R}_{(p)})$$

for $\mathcal{G} = G$ or $G \times \mathbb{C}^*$, as in the case of $\mathcal{R}$.

Away from 0: fibers of $\mathcal{R}_{(p)} \subset \mathcal{T}_{(p)}$ are identified with $\mathcal{R}^p \subset \mathcal{T}^p$. The action of G is identified with the diagonal action. $\mathbb{C}^*$ does not act, but $\mu_{\mathrm{p}} \subset \mathbb{C}^*$ does act an its action is identified with the cyclic action. Therefore we have:

$$H^{BM,G\times\mathbb{C}^*}_{*-2\dim_{Gr^o_{(p)}}(\mathcal{T}^o_{(p)})}(\mathcal{R}^o_{(p)}) = H^{BM,G\times\mu_{\mathrm{p}}}_{*+2-2\dim_{Gr^p}(\mathcal{T}^p)}(\mathcal{R}^p).$$

So by using Steenrod's construction we can define a non-linear graded map

$$A^{(1)} \to H^{BM,G^p\rtimes\mu_{\mathrm{p}}}_{*-2\dim_{Gr^p}(\mathcal{T}^p)}(\mathcal{R}^p) \to H^{BM,G\times\mu_{\mathrm{p}}}_{*-2\dim_{Gr^p}(\mathcal{T}^p)}(\mathcal{R}^p) = H^{BM,G\times\mathbb{C}^*}_{*-2-2\dim_{Gr^o_{(p)}}(\mathcal{T}^o_{(p)})}(\mathcal{R}^o_{(p)}).$$

We will compose this with the restriction

$$H^{BM,G\times\mathbb{C}^*}_{*-2-2\dim_{Gr^o_{(p)}}(\mathcal{T}^o_{(p)})}(\mathcal{R}^o_{(p)}) \to H^{BM,G\times\mu_{\mathrm{p}}}_{*-2-2\dim_{Gr^o_{(p)}}(\mathcal{T}^o_{(p)})}(\mathcal{R}^o_{(p)}).$$

Since $G \times \mu_{\mathrm{p}}$ acts trivially on the base $\mathbb{G}_a$, we are able to apply *specialization* in Borel-Moore homology to get a map

$$H^{BM,G\times\mu_{\mathrm{p}}}_{*-2-2\dim_{Gr^o_{(p)}}(\mathcal{T}^o_{(p)})}(\mathcal{R}^o_{(p)}) \to H^{BM,G\times\mu_{\mathrm{p}}}_{*-2\dim_{(Gr_{(p)})_0}((\mathcal{T}_{(p)})_0)}((\mathcal{R}_{(p)})_0).$$

Over 0: $\mathcal{R}_{(p)} \subset \mathcal{T}_{(p)}$ is identified with $\mathcal{R} \subset \mathcal{T}$. The action of $G \times \mathbb{C}^*$ is its usual action. Thus we have produced a map of graded sets:

$$A^{(1)} \to H^{BM, G\times\mu_p}_{*-2\dim_{(Gr_{(p)})_0}((\mathcal{T}_{(p)})_0)}((\mathcal{R}_{(p)})_0) = H^{BM, G\times\mu_p}_{*-2\dim_{Gr}(\mathcal{T})}(\mathcal{R}).$$

But we have $H^{BM, G\times\mu_p}_{*-2\dim_{Gr}(\mathcal{T})}(\mathcal{R}) = A_\hbar[a]$, so for degree reasons we produced a graded map of sets

$$F_\hbar : A^{(1)} \to A_\hbar.$$

This map is linear, because the averaging map $A \to A_\hbar[a]$ equals 0.

Fact 5.6. $F_\hbar$ gives the structure of Frobenius-constant quantization.

Example 5.7. $G = \mathbb{C}^*$, $\mathbf{N} = \mathbb{C}_{-r}$, $r \geqslant 0$. Then $F_\hbar$ is the map induced by usual Frobenius-constant structure on the Weyl algebra

$$\begin{aligned} x &\mapsto x^p \\ y &\mapsto \partial^p \end{aligned}$$

It must be therefore that

$$AS_\hbar((rxy)^r x) \in A_\hbar.$$

If so then it is automatically central. We can check

$$AS_\hbar((rxy)^r x) = \prod_{i=1}^{p}(rx\partial - ri\hbar)^r x^p = \prod_{i=1}^{p}(rx\partial - i\hbar)^r x^p = \prod_{i=1}^{pr}(rx\partial - i\hbar)x^p.$$

Remark 5.8. Can turn this calculation into a proof for large p by torus localization. But, it is true for all p, by direct geometric argument.

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